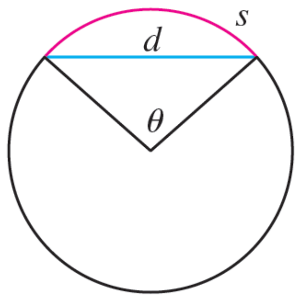
Exercise 57

The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find

$$\lim_{\theta\to 0^+} \frac{s}{d}$$



Solution

Use the law of cosines to write d in terms of θ. Let r be the radius of the circle.

$$\begin{aligned} d^2 &= r^2 + r^2 - 2(r)(r)\cos\theta \\ &= 2r^2 - 2r^2\cos\theta \\ &= 2r^2(1-\cos\theta) \end{aligned}$$

Solve for d.

$$d = r\sqrt{2(1-\cos\theta)}$$

The arc length for a circle is just $s = r\theta$. Now calculate the desired limit by writing it in terms of a known limit.

$$\begin{aligned} \lim_{\theta\to 0^+} \frac{s}{d} = \lim_{\theta\to 0^+} \frac{r\theta}{r\sqrt{2(1-\cos\theta)}} &= \lim_{\theta\to 0^+} \frac{\theta}{\sqrt{2(1-\cos\theta)}} \\ &= \lim_{\theta\to 0^+} \frac{\theta}{2\sqrt{\frac{1-\cos\theta}{2}}} \\ &= \lim_{\theta\to 0^+} \frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ &= \lim_{\alpha\to 0^+} \frac{\alpha}{\sin\alpha} \\ &= \lim_{\alpha\to 0^+} \frac{1}{\frac{\sin\alpha}{\alpha}} \\ &= \frac{1}{\displaystyle\lim_{\alpha\to 0^+} \frac{\sin\alpha}{\alpha}} \\ &= 1 \end{aligned}$$